Perturbative corrections to geodesic motion on soliton moduli spaces

Bruno Barton-Singer

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FOUNDATION FOR RESEARCH AND TECHNOLOGY - HELLAS

Work with Stavros Komineas and Grigorios Fournodavlos

Introduction: the geodesic approximation and when it fails

Finding perturbative corrections to geodesic motion

- General method
- Examples: the travelling kink and the collapsing lump

Finding perturbative corrections to geodesic + potential motion

- Method
- Examples: the reflected kink and the rebounding skyrmion

Conclusions and outlook

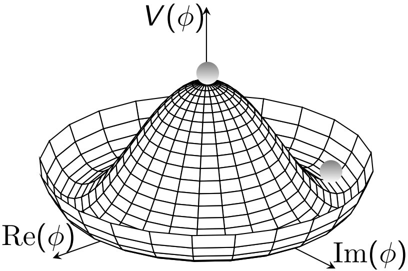
The geodesic approximation

- We have some (possibly reduced) action of the form $S(\phi) = \int \frac{1}{2}\dot{\phi}^2 d^D x dt - E(\phi)$.
- It has a continuous set of static solutions $\chi(\vec{x};\vec{c})$ given by $\frac{\delta E}{\delta \phi}\big|_{\chi} = 0$.
- It follows that $E(\chi) = E_0$, a constant independent of $\vec{c}$.
- We suppose we can approximate dynamical solutions to the action by $\phi(\vec{x},t) = \chi(\vec{x};\vec{c}(t))$, provided $\dot{\vec{c}}$ is sufficiently small. These $\vec{c}$ are called collective co-ordinates, hence 'collective co-ordinate approximation'.
- Substituting this solution into the action, we get an effective action: $S_{\text{eff}}(\vec{c}) = g_{ij}(\vec{c})\dot{c}_i\dot{c}_j - E_0$, where $g_{ij}(\vec{c}) = \int \frac{1}{2}\partial_{c_i}\chi\partial_{c_j}\chi d^D x$.
- This has dynamical equations $\frac{d}{dt}\left(g_{ij}(\vec{c})\dot{c}_j\right) = 0$; geodesics on the moduli space

Intuitive picture: the Higgs Bobsleigh

Why did we (Nick) come up with this approximation?

- For intuition, consider a finite-dimensional model: a point moving in the Higgs potential.
- Our 'field' is just a complex number: $\phi = \phi_1 + i\phi_2$
- Our energy is $E(\phi) = \frac{\lambda}{4}(|\phi|^2 - h^2)^2$; 'wine bottle'.



- Circle of static solutions: $\chi(c) = he^{ic}$.
- We now look for an approximate dynamical solution of the form $\chi(c(t))$. Implementing the procedure, we find $\dot{c} = \text{const}$.

Intuitive picture: the Higgs bobsleigh

- What about an exact solution? First rewrite ϕ in terms of $r = |\phi|$, $\alpha = \arg(\phi)$.
- $S(\phi) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\alpha}^2) - \frac{\lambda}{4}(r^2 - h^2)^2$
- Euler-Lagrange equations:

$$-\ddot{r} + r\dot{\alpha}^2 - \lambda r(r^2 - h^2) = 0$$
$$\frac{d}{dt}(r^2\dot{\alpha}) = 0 \implies r^2\dot{\alpha} = L.$$

- Looking for a solution where $\ddot{r} = 0$, we find $\lambda r^4(r^2 - h^2) = L^2$: solutions exist, with $r(t) = h + O(L^2)$.
- We see that the true solution is arbitrarily close to the geodesic approximation, provided we choose sufficiently small non-zero $\dot{c}$.
- Note: the true solution is equal to the approximate solution plus a small component orthogonal to the moduli space.

Failure of the geodesic approximation: collapse of a sigma-model lump

Let us now consider axisymmetric solutions of the $O(3)$ sigma model, described by a field $\Theta(r)$ with energy functional

$$E(\Theta) = \int_0^\infty \left(\frac{1}{2}\Theta'^2 + \frac{1}{2r^2}\sin^2\Theta \right) rdr$$

- Solutions given by $\chi(r;c) = 2\arctan\left(\frac{c^k}{r^k}\right)$, k integer.
- Geodesic approximation: $g(c) = \int (\partial_c \chi)^2 rdr$: changing variable to $r' = r/c$, we see that $g(c)$ is a constant*.
- It follows that one solution of the geodesic equation is $c(t) = v(t^\star - t)$; linear collapse.
- From numerics, we find that real collapse is slower: $\dot{c} \to 0$ as $c \to 0$.

Motivation: understand collapse of an antiferromagnetic skyrmion

- Chiral antiferromagnet model: described by Néel vector, $\boldsymbol{n}(\vec{x})$ (staggered magnetisation).
- Static energy functional basically same as ferromagnet:

$$E(\boldsymbol{n}) = \int \frac{1}{2}\partial_i \boldsymbol{n} \cdot \partial_i \boldsymbol{n} + \boldsymbol{D}_i \cdot (\boldsymbol{n} \times \partial_i \boldsymbol{n}) + h(1 - n_3^2) d^2x$$

  but unlike ferromagnets, dynamics is second-order.
- Looking at the axisymmetric theory $\boldsymbol{D}_i = -k\mathbf{e}_i$ and looking for axisymmetric solutions, we have

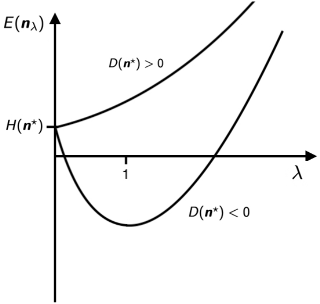
$$E = \int_0^\infty \left(\frac{1}{2}\Theta'^2 + \frac{1}{2r^2}\sin^2\Theta - \frac{2k}{r}\sin\Theta\sin^2\frac{\Theta}{2} + h\sin^2\Theta \right) rdr + b.t$$

Derrick scaling of antiferromagnetic skyrmion

Given a static solution $\boldsymbol{n}^\star(\vec{x})$, Derrick scaling tells us

$$E(\boldsymbol{n}^\star(\lambda\vec{x})) = H(\boldsymbol{n}^\star) - k\lambda D(\boldsymbol{n}^\star) + h\lambda^2 U(\boldsymbol{n}^\star)$$

Solitons can exist, but they only have a finite barrier to collapse:



Collective co-ordinate approximation: $S(\lambda) \sim \frac{1}{2}\dot{\lambda}^2 - (-k\lambda + h\lambda^2)$.
If initial kinetic energy is high enough, we get linear collapse. If it is smaller, the skyrmion rebounds and oscillates.
But numerics does not agree...

Sketch of approach for any 1D moduli space

This is inspired by Bizoń, Ovchinnikov and Sigal's calculation for the collapse of an instanton - same core idea, but it relied on modulus being a scaling symmetry so it could be absorbed into co-ordinates. Clearer to present general version.

- We look for a solution of the form $\phi(x,t) = \chi(x; c(t)) + w(x, c, \dot{c})$
  - w 'orthogonal' to moduli space: any perturbation that overlaps with moduli space could be included by changing c.
  - By assuming w has no *explicit* time dependence, we are not considering ingoing or outgoing radiation.
- Substitute in Euler-Lagrange equation to find equation relating $\ddot{c}$, $\dot{c}$, c and w.
  - We can split the E-L into a part parallel to the moduli space, and a part perpendicular.
  - The perpendicular part can in principle be solved for w - then the parallel part gives a corrected ODE for $c(t)$. (L-S)
- Then expand both $\ddot{c}$ and w in powers of $\dot{c}$.
  - Truncating at $O(\dot{c}^2)$, we recover geodesic equation for $c(t)$.
  - Quartic order gives first non-trivial correction.

Step 1: Lyapunov-Schmidt decomposition

- Euler-Lagrange equation: $\partial_t^2 \phi - L(\phi) = 0$
- After substituting our ansatz:
$\dot{c}^2 \partial_c^2 \chi + \ddot{c} \partial_c \chi + \underbrace{L_\chi}_{\text{Hessian}} w + \underbrace{N(w; x; c)}_{\text{nonlinearity}} + \partial_t^2 w = 0$
- Let us introduce $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx$ (or $\int_0^\infty f(r) g(r) r dr$..)
- $g(c) = \langle \partial_c \chi, \partial_c \chi \rangle$. $g'(c) = 2 \langle \partial_c \chi, \partial_c^2 \chi \rangle$. $\langle \partial_c \chi, L_\chi w \rangle = 0$.

Parallel

$$\dot{c}^2 \langle \partial_c \chi, \partial_c^2 \chi \rangle + \ddot{c} \langle \partial_c \chi, \partial_c \chi \rangle + \langle \partial_c \chi, N \rangle + \langle \partial_c \chi, \partial_t^2 w \rangle = 0$$

Perpendicular

$$\dot{c}^2 (\partial_c^2 \chi)_\perp + L_\chi w + N_\perp + (\partial_t^2 w)_\perp = 0$$

First two terms of first equation: geodesic equation for $c(t)$.

Step 2: expansion in powers of $\dot{c}$

Due to $t \to -t$ symmetry, we know that $\ddot{c}$ and w are even functions of $\dot{c}$.

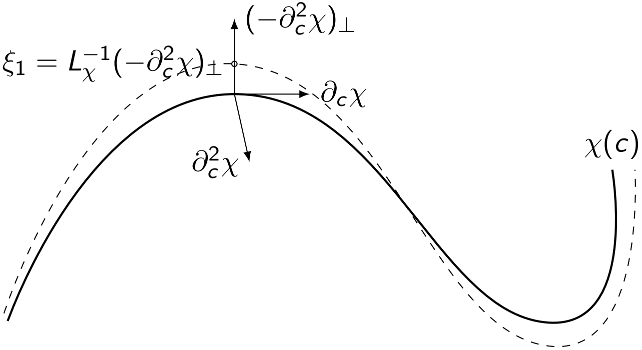
- $\ddot{c} = \sum_i f_i(c)\dot{c}^{2i}$
- $w = \sum_i \xi_i(x; c)\dot{c}^{2i}$

$\partial_t^2 w$ will have terms like $\partial_c \xi_i$, $\partial_c^2 \xi_i$ and $\partial_c \xi_i f_j$, $\xi_i f_j f_k$ - we can check that all of this enters at higher-order than $\dot{c}^{2i}$.
At $O(\dot{c}^2)$:

$$\langle \partial_c \chi, \partial_c^2 \chi \rangle + f_1(c)\langle \partial_c \chi, \partial_c \chi \rangle = 0 \qquad (\parallel \partial_c \chi)$$
$$(\partial_c^2 \chi)_\perp + L_\chi \xi_1 = 0 \qquad (\perp \partial_c \chi)$$

- Parallel part tells us $f_1(c) = -\frac{g'(c)}{2g(c)}$ - so that if we truncate the expansion for $\ddot{c}$ here, we get back the geodesic equation.
- But this geodesic motion generates a perturbation ξ_1, which will enter into the parallel part at quartic order...
- You can check that this iterative procedure goes on forever; f_i and ξ_i are functions of $f_{j<i}$ and $\xi_{j<i}$.

Step back: geometric picture



- $(-\partial_c^2\chi)_\perp$ represents the direction of 'centrifugal force'
- L_χ gives the height of the valley around the moduli space
- $L_\chi^{-1}(-\partial_c^2\chi)_\perp$ gives direction of leading-order perturbation.

At next order we will see two effects:

- Interaction with cubic nonlinearity
- 'Figure skater' effect

Perturbative expansion: quartic order

Let us call the quadratic (in Euler-Lagrange) part of the nonlinearity $a_2(x;c)w^2$.
Parallel part, $O(\dot{c}^4)$:

$$\langle \partial_c \chi, \partial_c \chi \rangle f_2(c) + \langle \partial_c \chi, \partial_c^2 \xi_1 \rangle + 5\langle \partial_c \chi, \partial_c \xi_1 \rangle f_1 + \langle \partial_c \chi, a_2 \xi_1^2 \rangle = 0$$

Now what effect does this have on ODE?

$$\frac{d}{dt}\left(g(c)\dot{c}^2\right) = f_2(c)\dot{c}^5$$

Like a friction force, it goes to zero as the speed goes to zero. But it always acts in the same direction for a given sign of $f_2(c)$. Positive $f_2(c)$ slows down decreasing c and speeds up increasing c; negative $f_2(c)$ does the opposite.

Example 1a: the kink

$E(\phi) = \int \frac{1}{2}\phi'^2 + V(\phi) dx$

- In this model we can compare to exact dynamical solutions given by Lorentz-boosting the static solution.
- $\chi(x;c) = \chi_0(x-c)$ - somewhat trivial case where c can be absorbed into co-ordinates
- $g(c) = \int \chi'^2 dx$ is constant*, $g'(c) = 0 \implies f_1(c) = 0$
- Using the factorisation of L_χ due to the Bogomol'nyi argument, we can write the inversion explicitly: $L_\chi^{-1} f = \chi'(x-c) \int^x \frac{1}{\chi'(x'-c)^2} \int^{x'} \chi'(x''-c) f(x) dx''$
- We find $\xi_1(x;c) = \frac{1}{2} x \chi_0'(x-c)$.
- We also find $f_2(c) = 0$, and so $c = vt$ is still a valid solution.
- Lorentz boosted solution $\chi_0(\sqrt{1-v^2}(x-vt))$ can be expanded in powers of v, and prefactor of v^2 agrees with above.

Example 1b: collapsing lump

$$E(\Theta) = \int_0^\infty \left(\frac{1}{2}\Theta'^2 + \frac{1}{2r^2}\sin^2\Theta \right) rdr; \qquad \chi(r;c) = 2\arctan\left(\frac{c^k}{r^k}\right).$$

- As before, $g(c)$ constant, $g'(c) = 0$.
- Bogomol'nyi argument means L_χ can be factorised, meaning we can explicitly find ξ_1.
- We find $f_2(c) = \frac{3}{4c}$; our first correction to the geodesic equation is $\ddot{c} = \frac{3}{4c}\dot{c}^4$. (Numerical factor unimportant)
- Note that if we if we tried to take solution to geodesic equation with $c = \epsilon(t^\star - t)$ and add an ϵ^2 correction, it would not work.
- On integrating the above, we find $\dot{c} \sim \frac{1}{\sqrt{\log(c)}}$. Speed goes to zero as $c \to 0$, and behaviour matches numerics.

Finding perturbative corrections to collective co-ordinate approximation in presence of potential

For a perturbative expansion to make sense, our potential should go to zero as our speed goes to zero.
We assume $\dot{c}(t) = \epsilon \dot{y}(t)$ with $\dot{y}(t) = O(1)$, and then assume that our energy is modified from one with a moduli space of static solutions by a term of $O(\epsilon^2)$, leading to a change in the Euler-Lagrange equations $\epsilon^2 \delta L(\Theta)$.
We must rephrase expansion of w:

$$\ddot{c}(t) = \sum_{i \geq 1} f_i(c; \dot{y}) \epsilon^{2i} \tag{1}$$

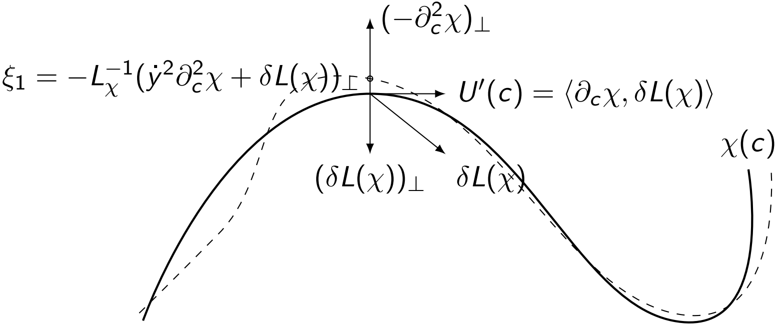
$$w = \sum_{i \geq 1} \xi_i(r; c, \dot{y}) \epsilon^{2i}. \tag{2}$$

At $O(\epsilon^2)$:

$$\langle \partial_c \chi, \partial_c^2 \chi \rangle \dot{y}^2 + \langle \partial_c \chi, \partial_c \chi \rangle f_1(c; \dot{y}) + \langle \partial_c \chi, \delta L(\chi) \rangle = 0 \tag{3}$$

$$(\partial_c^2 \chi)_\perp \dot{y}^2 + L_\chi \xi_1 + (\delta L(\chi))_\perp = 0 \tag{4}$$

Geometric picture



Now any change of combinations to the perpendicular force, nonlinearities or curvature can have effects at the next order.

Quartic order

$$f_1(c;\dot{y}) = -\frac{g'(c)}{2g(c)}\dot{y}^2 - \frac{U'(c)}{g(c)}$$
$$\xi_1 = -L_\chi^{-1}(\dot{y}^2\partial_c^2\chi + \delta L(\chi))_\perp$$

$$g(c)f_2(c,\dot{y}) + \langle \partial_c\chi, \partial_c^2\xi_1\rangle \dot{y}^2 + 2\dot{y}\langle \partial_c\chi, \partial_c\partial_{\dot{y}}\xi_1\rangle f_1 + \langle \partial_c\chi, \partial_c\xi_1\rangle f_1 + \langle \partial_c\chi, a_2\xi_1^2\rangle = 0.$$

We see that f_2 will have terms that are constant, quadratic and quartic in $\dot{y}$, each with their own interpretation:

$$f_2(c,\dot{y}) = \underbrace{f_{20}(c)}_{\text{corrects } U(c)} + \underbrace{f_{21}(c)\dot{y}^2}_{\text{corrects metric}} + \underbrace{f_{22}(c)\dot{y}^4}_{\text{energy source/sink}} \tag{6}$$

Example 2a: the kink in presence of delta-function potential

$$E(\Theta) = \int \frac{1}{2}\Theta'^2 + V(\Theta) + \epsilon^2 h\delta(x)V(\Theta)dx.$$

The effective potential on the moduli space is $U(c) = V(\chi(-c))$
Will $O(\epsilon^4)$ corrections have the kink reflect when it should pass, or vice versa?

$$L_\chi^{-1}(\delta L)_\perp = h\chi'(x-c)\underbrace{\chi'(-c)\chi''(-c)}_{2V(\chi(-c))'}\int^x \frac{1 - \int^{x'}\chi'(x''-c)^2 dx''/g}{\chi'(x'-c)^2}dx'$$

We find modifications to the potential $\delta U(c)$ which all feature $\chi''(-c)$ - i.e. the potential does not change at the maximum, where it matters.
So a single δ-function potential behaves as you'd expect from CCM - but two at an appropriate distance will not. Spectral walls?

Example 2b: the collapsing antiferromagnetic skyrmion (in progress)

There's a catch: the model with potential that we want to expand about is exactly that for which metric diverges (lump with $k = 1$.) Putting aside concerns about a divergent metric, for the antiferromagnetic skyrmion we get:

$$\ddot{c} = \underbrace{-k + h_a c}_{O(\epsilon^2)} + \underbrace{\frac{\dot{c}^4}{c}}_{O(\epsilon^4/c)} + O(\epsilon^4) \tag{7}$$

Three regimes:

- $|\dot{c}(0)| < v^\star$: periodic motion
- $|\dot{c}(0)| = v^\star$: collapse with behaviour $c \to \frac{k}{2}(t^\star - t)^2$
- $|\dot{c}(0)| > v^\star$: $\dot{c} \to \frac{1}{\sqrt{\log(t^\star - t)}}$

However, we know that in the case without potential, the divergent metric leads to a different behaviour. Dynamic cutoff? Expand near model with exponential tails?

Conclusions

- We can include the effects of perturbations off the moduli space as extra terms in the ODE, that are quartic, sextic etc. in the velocity of the moduli
- Sometimes the perturbations have a dramatic effect on solutions, cf. the collapsing lump.
- This method can be extended to include moduli spaces with potential, provided we know what exact theory we are expanding around
- The resulting collapse of an antiferromagnetic skyrmion seems to be fundamentally the same as that of the $O(3)$ sigma model - but further work needed.

Outlook

- Higher powers of $\dot{c}$
- Colliding kinks
- Divergent metrics
- Higher-dimensional moduli space, target space, domain
- Spectral walls?
- Small amplitude oscillations around a single static solution